Optimal Control of a Multi-field Irrigation Problem: validation of a numerical solution by the optimality conditions

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Abstract— **In a previous study, the authors developed an optimal control model to minimize the water flowing into a reservoir that supplies different fields with different types of crops, and where the water from the precipitation can be collected. In this paper, we validate the numerical solution obtained in a previous study verifying that the solution satisfies the necessary conditions of optimality in the form of a Maximum Principle.**

Keywords: Irrigation, Optimal Control, Maximum Principle.

I. Introduction

We aim to minimize the total of water flow coming from the tap to a reservoir,

$$\min \int_0^T v(t)dt,$$

where $v(t)$ denotes the total water flow coming from the tap at time t. The reservoir supplies the multiple cultivated fields where each field can have different areas and different cultures. The water from the precipitation also can be collected into the reservoir. Therefore, the variation of water in reservoir is given by

$$\dot{y}(t) = v(t) - \sum_{j=1}^{P} A_j u_j(t) + Cg(t),$$

where y represents total of amount of water stored in the reservoir, A_j represents the area of each field j, $Cg(t)$ represents collected water in a certain area C coming from the precipitation g in the time t, and u_j is the water flow introduced in field j via its irrigation system.

The variation of water in the soil is given by the hydrological balance equation, that is, the variation of water in the soil is equal to what enters (water via irrigation systems and precipitation) minus the loss (evapotranspiration of each crop h_j and loss by deep percolation $\beta x_j(t)$, a percentage of water that is in the soil). So,

$$\dot{x}_j(t) = u_j + g(t) - h_j(t) - \beta x_j(t), \forall j = 1, \ldots, P$$

where x_j water in the soil of field j.

We assume that each field has only one crop. In order to ensure that the crop is in good state of conservation, the water in the each field has to be sufficient to satisfy the hydric needs of each crop ($x_{\min}$), that is:

$$x_j(t) \geq x_{\min_j}.$$

The physical limitations of the amount of water that comes from a tap, the amount of water that comes from the irrigation systems, and the reservoir are given, respectively, by:

$$\begin{array}{ll} y(t) & \in [0, y_{\max}] \\ u_j(t) & \in [0, M_j] \\ v(t) & \in [0, \sum_j A_j M_j] \end{array}$$

where $y_{\max}$ is the maximum quantity of water in the reservoir and M_j is the maximum water flow coming from the tap in each field.

We assume that at the initial time the humidity in the soil of each field and the water in the reservoir are given. Also, the water in the reservoir at the initial time and to the final time are imposed to be equal. So, we assume that

$$\begin{array}{l} x_j(0) = x_{0_j} \\ y(0) = y(T) = y_0 \end{array}$$

In summary, the optimal control formulation for our problem (OCP) with P fields and final time T is

$$\min \int_0^T v(t)dt$$

subject to:

$$\dot{x}_j(t) = -\beta x_j(t) + u_j + g(t) - h_j(t) \quad \text{a.e. } t \in [0,T], \ \forall j = 1, \ldots, P$$

$$\dot{y}(t) = v(t) - \textstyle\sum_{j=1}^{P} A_j u_j(t) + Cg(t), \quad \text{a.e. } t \in [0,T],$$

$$x_j(t) \geq x_{\min_j}, \quad \forall t \in [0,T], \ \forall j = 1, \ldots, P$$

$$y(t) \in [0, y_{\max}], \quad \forall t \in [0,T]$$

$$u_j(t) \in [0, M_j], \quad \text{a.e. }, \forall j = 1, \ldots, P$$

$$v(t) \in [0, \textstyle\sum_j A_j M_j], \quad \text{a.e.}$$

$$x_j(0) = x_{0_j}, \quad \forall j = 1, \ldots, P$$

$$y(0) = y(T) = y_0.$$

II. Numerical Results

We use a direct method discretization to obtain the numerical results for the optimal control problem written as a mathematical programming problem. All parameters used in this simulation are defined as in [3]. Here, we also consider three crops: wheat in 1000 m^2, sugar cane in 750 m^2 and olive in 1250 m^2 and precipitation water is collected in area

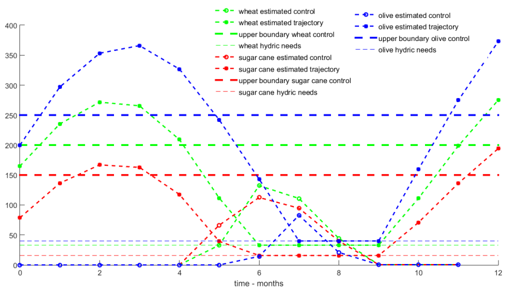

Fig. 1. Crops need.

200 m^2. The results obtained are reported in Fig. 1 and 2, (see also [3]):

As expected, the crops need water between May and September. The months when the water consumption is higher are June, July. The crop that needs less water is olive with the largest area 1250 m^2.

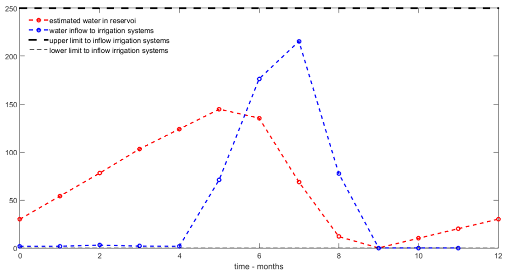

Fig. 2. Reservoir

We note that the water in the reservoir at the initial time is equal to the water at the final time. We can see that until April the water from the precipitation is saved in the reservoir and until this month the tap is not opened. The tap is closed in September. June and July are the months when the irrigation takes the highest value, and also when the water in the reservoir decrease.

III. Validation of the Solution

Since our problem has inequality constraints that are active for some period of time, it is not easier to get an analytical solution. However, we can verify that the numerical solution for the mathematical programming problem satisfy the necessary condition of optimality in the form of Maximum Principle (MP), namely the Weierstrass Condition. Here, we assume that the constraint qualifications introduced in [2] or [1] is satisfied, and we apply Weierstrass Condition of the MP in the normal form, see for example [4], and we obtain:

$$(\mathbf{q}(t) - \mathbf{A}) \cdot (\mathbf{u} - \bar{\mathbf{u}}) + (w(t) - 1)(v(t) - \bar{v}(t)) \leq 0 \quad (1)$$

Assuming that $(\bar{u}, \bar{v}) \in]0, \mathbf{M}[\times]0, R[$, the Weierstrass Condition can be written as:

$$\begin{cases} \mathbf{q}(t) = \mathbf{A}(t) \\ w(t) = 1. \end{cases}$$

From Fig. 1 and Fig. 2, we can conclude that $(\bar{u}, \bar{v}) \in]0, \mathbf{M}[\times]0, R[$ in May, June, July and August, and therefore we can apply the Weierstrass Condition equation above in theses months.

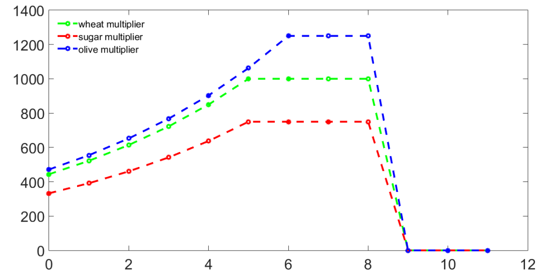

Fig. 3. Multipliers associated to the crops

We can observe the multiplier associate to the crops state are equal to the area of the fields of the corresponding crops in the months mentioned above:

$$\begin{array}{ll} \text{wheat multiplier} := & 1000 \\ \text{sugar canne multiplier} := & 750 \\ \text{olive multiplier} := & 1250. \end{array}$$

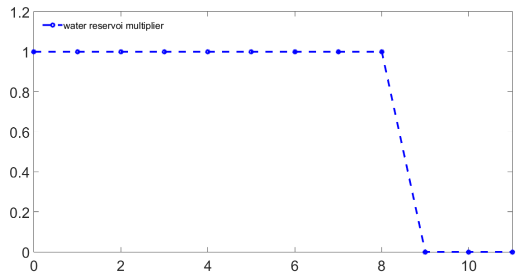

Fig. 4. Multipliers associated to the water in the reservoir

We also can see that in the previously mentioned months the multiplier associate to reservoir state is equal to 1.

We conclude that for $(\bar{u}, \bar{v}) \in]0, \mathbf{M}[\times]0, R[$ the numerical solution fulfills the Weierstrass Condition of the Maximum Principle in the normal form.

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